Construction of Lebesgue Measure

**Motivation.** What are length, area and volume?What is a quantity of 4 or more dimensional object?

**Question.** How can we define quantity measurement of some sets in Euclidean space?

**Definition1.** Let with .

Define naturally the length of each interval such that

*, , , ,*

and where are disjoint.

**Definition 2.**

(1). A family of subsets of a set is said to be an ‘algebra’if it satisfies followings.

(2).A family of subset of a set is said to be a ‘-algebra’ if it satisfies followings.

**Definition 3.** Let be an algebra and be a-algebra of a set

(1). A ‘measure’ on is an extended real-valued function defined on such that

1. If is a disjoint sequence in with, then .

(2). A ‘measure’ on is an extended real-valued function defined on such that

1. for any disjoint sequence in

**Lemma 1.** Let be a collection of all finite union of set of the form

Then is an algebra of subsets of and length is a measure on .

**Proof)**By construction of , it is readily seen that is an algebra. And satisfies the condition I and II of Definition 3 trivially.

It is enough to show that where are disjoint implies to check the condition IIIof Definition 3.

Let where are disjoint and consider any finite collection of such intervals.

Suppose that

by renumbering the finite indices.

Then

Since is arbitrary, we infer that

Conversely, let and be a sequence of positive numbers with *.*

Consider the intervals for .

Since for sufficiently small , is an open covering of the compact set . It implies that there is a finite sub-covering of . By renumbering and discarding some extra intervals, we may assume that

Then

Thus we get

Since is arbitrary, it follows that .

Therefore, we get .

**Definition 4.** Let be an arbitrary subset. Define

the ‘outer measure’ generated by withan algebra .

**Lemma 2.** The function of Definition 4 satisfies the followings.

1. If , then
2. If , then
3. If is a sequence of subsets of , then

**Proof)**(a), (b) and (c) are trivial.

(d): Since is a countable cover of B in , it follows that

Conversely, if is a sequence in whose union contains , then

Since is a measure on , by II andIIIof definition 3, we have known that

Since is arbitrary chosen, it follows that .

(e): Let is a sequence of subsets of and *.*Choose a sequence in for each such that

Since is a countable collection in such that , it follows from the definition of that

Since is arbitrary chosen, we get .

**Definition 5. [*Carathéodory Condition*]**

A subset of is said to be ‘-measurable’ if

Define*.*

**Theorem 1. [*Carathéodory Extension Theorem*]**

The collection is a -algebra containing .Moreover, if is a disjoint sequence in , then

**Proof)**Clearly, and implies .

Claim: is closed under intersections

Let . Then for any and , we have

Since , .

Let . Then it is readily seen that

It follows that

Thus, we get

Which shows that .

Since is closed under intersections and complementation, is an algebra.

Let with . Since and , we obtain

For , it shows that is finitely additive on .

Let be a disjoint sequence in and . Since is an algebra, we know that and finite additivity implies

for any and each .

Since , and letting implies

Conversely, it follows from Lemma 2. (e) that

From the last three inequalities, we get

This shows that . For , we get .

It remains to show that . Let and be arbitrary. By Lemma 2. (d) and (e), we know that

Let and be a sequence such that and

by definition of outer measure.

Since and , it follows from Lemma 2. (e) that

Hence we have

Since is arbitrary, . Therefore, we conclude that for any , that means.

**Conclusion.** Suppose the algebra defined in Lemma 1 and the ‘length’ that is a measure on .

Consider

,the outer measure generated by and

,the collection of subsets of that satisfies the Carathéodory Condition.

By Theorem 1, is a -algebra containing and is a measure on .

A subset of contained in is called a ‘Lebesgue measurable set’ and is called the ‘Lebesgue measure’.

**Theorem 2. [*Hahn Extension Theorem*]**

If is a measure on such that for all open intervals , then .In other word, the Lebesgue measure is unique.

**Proof)**For , let . Let be any set with for some and be a sequence of open intervals such that . Since is a measure and for all , we have

Therefore, for all Lebesgue measurable sets .

Since and are additive,

Since all of these terms are finite and and , it follows that for all Lebesgue measurable sets .

Let be an arbitrary Lebesgue measurable set. Then it can be written as the union of disjoint Lebesgue measurable sets , defined by

Since for all by previous step, it follows that

Therefore, on .

**Reference:***Bartle, The Elements of Integration and Lebesgue Measure, 1995*